

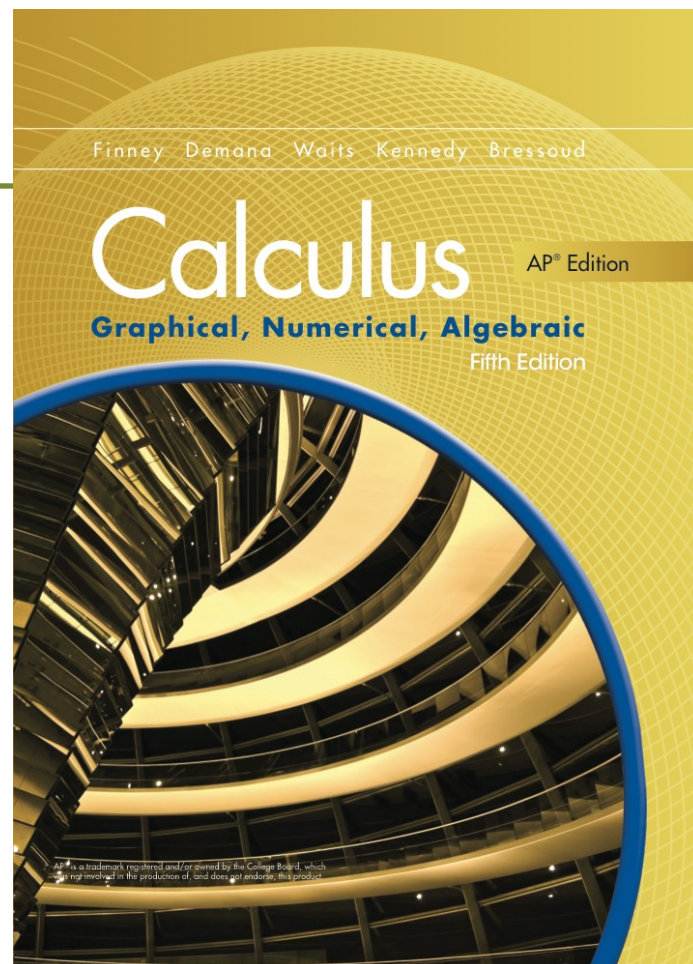
Chapter 4

More Derivatives



Section 4.1

Chain Rule



Quick Review

In Exercises 1–5, let $f(x) = \sin x$, $g(x) = x^2 + 1$, and $h(x) = 7x$.

Write a simplified expression for the composite function.

1. $f(g(x))$

2. $f(g(h(x)))$

3. $(g \circ h)(x)$

4. $(h \circ g)(x)$

5. $f\left(\frac{g(x)}{h(x)}\right)$

Quick Review

In Exercises 6–10, let $f(x) = \cos x$, $g(x) = \sqrt{x+2}$, and $h(x) = 3x^2$. Write the given function as a composite of two or more of f , g , and h . For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$

7. $\sqrt{3\cos^2 x + 2}$

8. $3 \cos x + 6$

9. $\cos 27x^4$

10. $\cos \sqrt{2 + 3x^2}$

Quick Review Solutions

In Exercises 1–5, let $f(x) = \sin x$, $g(x) = x^2 + 1$, and $h(x) = 7x$

Write a simplified expression for the composite function.

$$1. \quad f(g(x)) \quad \sin(x^2 + 1) \qquad 2. \quad f(g(h(x))) \quad \sin(49x^2 + 1)$$

$$3. \quad (g \circ h)(x) \quad 49x^2 + 1 \qquad 4. \quad (h \circ g)(x) \quad 7x^2 + 7$$

$$5. \quad f\left(\frac{g(x)}{h(x)}\right) \quad \sin\frac{(x^2 + 1)}{7x}$$

Quick Review Solutions

In Exercises 6-10, let $f(x) = \cos x$, $g(x) = \sqrt{x+2}$, and $h(x) = 3x^2$. Write the given function as a composite of two or more of f , g , and h . For example, $\cos 3x^2$ is $f(h(x))$.

6. $\sqrt{\cos x + 2}$ $g(f(x))$ 7. $\sqrt{3\cos^2 x + 2}$ $g(h(f(x)))$

8. $3\cos x + 6$ $h(g(f(x)))$ 9. $\cos 27x^4$ $f(h(h(x)))$

10. $\cos\sqrt{2+3x^2}$ $f(g(h(x)))$

What you'll learn about

- The Chain Rule for differentiating a composite function
- The Chain Rule in prime and Leibniz notations
- Differentiating parametrically defined functions
- The Power Chain Rule
- Using the Chain Rule to show how degree measure affects the calculus of trig functions

... and why

The chain rule is the most widely used differentiation rule in mathematics.

Rule 8 The Chain Rule

If f is differentiable at the point $u=g(x)$, and g is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibnitz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example Derivatives of Composite Functions

An object moves along the x - axis so that its position at any time $t \geq 0$ is given by $s(t) = \sin(5t - 3)$. Find the velocity of the object as a function of t

$s(t) = \sin(5t - 3)$ is a composite function: $s = \sin(u)$ and $u = 5t - 3$

$$\frac{ds}{du} = \cos(u) \quad \text{and} \quad \frac{du}{dt} = 5$$

By the Chain Rule
$$\frac{ds}{dt} = \frac{ds}{du} \cdot \frac{du}{dt}$$

$$\frac{ds}{dt} = \cos(u) \cdot 5$$

$$\frac{ds}{dt} = 5\cos(5t - 3)$$

“Outside-Inside” Rule

It sometimes helps to think about the Chain Rule this way:

$$\text{If } y = f(g(x)), \text{ then } \frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the “outside” function f and evaluate it at the “inside” function $g(x)$ left alone; then multiply by the derivative of the “inside” function.

Example “Outside-Inside” Rule

Differentiate $\cos(3x^4 - 2)$ with respect to x

$$\begin{aligned} \frac{d}{dx} \cos(\underbrace{3x^4 - 2}_{\text{inside}}) &= - \sin(\underbrace{3x^4 - 2}_{\text{inside left alone}}) \cdot \underbrace{12x^3}_{\text{derivative of inside}} \\ &= -12x^3 \sin(3x^4 - 2) \end{aligned}$$

Example Repeated Use of the Chain Rule

Sometimes the chain rule needs to be used more than once to find a derivative.

Find the derivative of $y = \sqrt[3]{\sin x^2}$.

$$y = \sqrt[3]{\sin x^2} = (\sin x^2)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3} (\sin x^2)^{-\frac{2}{3}} (\cos x^2) \cdot 2x$$

$$\frac{dy}{dx} = \frac{2}{3} x (\sin x^2)^{-\frac{2}{3}} (\cos x^2)$$

Slopes of Parametrized Curves

A parametrized curve $(x(t), y(t))$ is *differentiable at t* if x and y are differentiable at t .

Finding dy/dx Parametrically

If all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Power Chain Rule

If f is a differentiable function of u , and u is a differentiable function of x , then substituting $y = f(u)$ into the Chain Rule

formula $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ leads to the formula

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$